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Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation

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In this paper we show that the discrete Nagumo equation

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z},$$

has a traveling wave solution for sufficiently strong coupling d.

The problem is at first simplified into a fixed point problem which can be solved by Brouwer's fixed point theorem. The solutions of the simplified problem are then continued via index-theory to solutions of approximate problems. In the final step it is proven that the solutions of the approximate problems have a limit point which corresponds to a solution of the original problem. © 1992 Academic Press, Inc.

1. INTRODUCTION

Consider the infinite system of coupled nonlinear differential equations

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z}, \tag{1}$$

where d is a positive real number and f denotes a Lipschitz continuous function satisfying

$$\begin{gathered} f(0) = f(a) = f(1), \qquad f(x) < 0, \qquad \text{for} \quad 0 < x < a, \\ f(x) > 0, \qquad \text{for} \quad a < x < 1, \quad \text{and} \quad \int_0^1 f(x)\, dx > 0. \end{gathered} \tag{2}$$

A typical example is the cubic polynomial

$$f(x) = x(x-a)(1-x), \qquad \text{where} \quad 0 < a < \tfrac{1}{2}.$$

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Equation (1) is interesting for several reasons. First, it is the discrete analogue to the well-known Nagumo equation [9],

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u). \tag{3}$$

The Nagumo equation is used as a model for the spread of genetic traits [2] and for the propagation of nerve pulses in a nerve axon, neglecting recovery [8, 10]. The discrete Nagumo equation (1) has been used to derive Eq. (3) [14]. It has also been proposed as a model for conduction in myelinated nerve axons [3]. The continuous Nagumo equation (3) is well studied [7]. It has been shown, for example, that there is a function U, with $U(-\infty)=0$, $U(\infty)=1$, and a constant $c>0$ such that

$$u(x, t) := U\left(\frac{x}{\sqrt{D}} + ct\right)$$

is a solution of (3) for all $D>0$. Such solutions are called traveling wavefronts, or simply, traveling waves.

In the discrete Nagumo equation, a similar result cannot be true for small d [12]. This difference between the continuous and the discrete model has led, for instance, to new insights regarding the formation of spiral waves [13].

There are already some results known for the discrete Nagumo equation. The first results were concerned with threshold properties of the system, in particular, with conditions forcing nonconvergence to zero of solutions as time approaches infinity and with bounds on the speed of propagation of a "wave of excitation" [3, 4]. It has been conjectured that there are traveling wave solutions of (1), i.e., solutions of the form

$$u_n(t) = U(n + ct)$$

with $U(-\infty)=0$ and $U(\infty)=1$. Numerically, such solutions were found and analyzed for certain cubic polynomials f [6]. In [5], a first attempt was made to prove the existence of traveling wave solutions; unfortunately the proof given in [5] was not conclusive. It is the aim of this paper to give a rigorous proof of the conjectured existence of traveling wave solutions for sufficiently large d. The precise statement of the result is:

THEOREM 1. *Suppose f is a Lipschitz continuous function satisfying (2). Then there exists some $d^*>0$ such that for $d>d^*$ the discrete Nagumo equation* (1) *admits a solution $u_n(t) = U(n+ct)$, where $c>0$, $U \in C^1(\mathbb{R}, (0, 1))$, $U(-\infty)=0$, $U(\infty)=1$, and $U'(x)>0$ for all $x \in \mathbb{R}$.*

The proof of Theorem 1 can be broken into four steps.

Step 1. The problem is simplified.

Instead of the discrete Nagumo equation we consider

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\ u_n &= P(v_n), \qquad n \in \mathbb{Z}, \end{aligned} \tag{4}$$

where

$$h(u_n) := u_n - \tfrac{1}{4}$$

and

$$P(v_n) := \begin{cases} 0 & \text{for} \quad v_n < 0 \\ v_n & \text{for} \quad 0 \leqslant v_n \leqslant 1. \\ 1 & \text{for} \quad 1 < v_n. \end{cases}$$

The nonlinear function f has been replaced by the affine function h (see Fig. 1) and then u_n is forced to take values in the interval $[0, 1]$ via the Lipschitz continuous projection P: $\mathbb{R} \to [0, 1]$ (see Fig. 2).

The *simplified problem* is to find a monotone traveling wave of (4), i.e., a solution of (4) on an interval $[0, \tau]$ which satisfies the conditions

(C1) $u_n(\tau) = u_{n+1}(0)$

(C2) $v_n(0) \leqslant v_{n+1}(0)$,

(C3) $d(u_{n-1}(0) - 2u_n(0) + u_{n+1}(0)) + h(u_n(0)) > 0$ if $u_n(0) > 0$,

(C4) $\lim_{n \to -\infty} v_n(0) = 0$, and $\lim_{n \to \infty} v_n(0) = 1$.

Note that conditions (C1), (C2), and (C4) imply the existence of a function U: $\mathbb{R} \to \mathbb{R}$ with $U(-\infty) = 0$, $U(\infty) = 1$, $0 \leqslant U \leqslant 1$, such that

$$u_n(t) = U(n + ct), \qquad \tau = 1/c, \qquad \text{for all} \quad n \in \mathbb{Z},\ t \in [0, \tau].$$

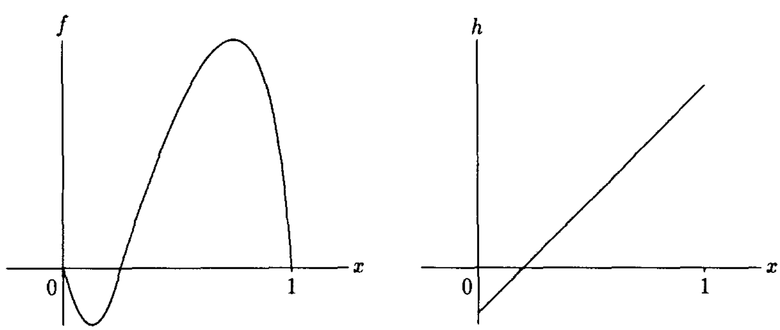

FIG. 1. The nonlinear function f and its simplification h.

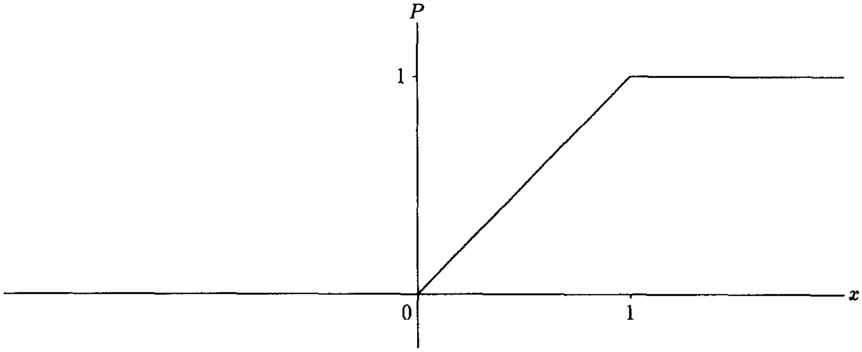

FIG. 2. The Lipschitz continuous projection P.

It will be shown (cf. Lemma 5) that for a monotone traveling wave only finitely many of the values $u_n(0)$ are different from 0 and 1 (see Fig. 3).

It suffices therefore to consider only a finite number of the equations (4).

Step 2. The simplified problem is converted into a fixed point problem.

Note that the initial value problem

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\ u_n &= P(v_n), \qquad\qquad\qquad\qquad\qquad\qquad\qquad (5) \\ v_n(0) &= x_n \qquad \text{with} \quad 0 \leqslant x_n \leqslant 1 \text{ for } n = 0, ..., N, \end{aligned}$$

where $u_{-1} \equiv 0$ and $u_{N+1} \equiv 1$, has a unique solution, say

$$u(x; t) = \{u_n(x; t)\}_{n=0}^{N},$$

which depends continuously on the initial value $x = \{x_n\}_{n=0}^{N}$. For an appropriate initial value space X we will then consider the following shifted Poincaré map

$$T: \bar{X} \to \mathbb{R}^{N+1},$$

$$(Tx)_n := \begin{cases} 0 & \text{for} \quad n = 0 \\ u_{n-1}(x; \tau) & \text{for} \quad n = 1, ..., N, \end{cases}$$

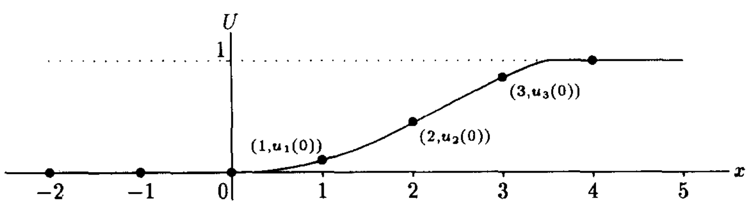

FIG. 3. A monotone traveling wave.

where τ is defined implicitly by

$$u_0(x; \tau) = x_1.$$

As an appropriate definition of X we will take

$$X := \Big\{ \{x_n\}_{n=0}^{N} \in \mathbb{R}^{N+1} : x_0 = 0,\ x_1 = -h(0)/d,$$
$$x_n \leqslant x_{n+1},\ d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0,$$
$$x_n \geqslant \frac{n}{N}, \text{ for } n = 1, ..., N, \text{ where } x_{N+1} := 1 \Big\}. \tag{6}$$

Note that X depends on d and N. It turns out that for a suitable choice of d and N the map T is well defined and continuous, $\bar{X}$ is nonempty and convex, and T maps $\bar{X}$ into X.

It follows then by Brouwer's fixed point theorem that T has a fixed point x. This fixed point x corresponds to the traveling wave $\{u_n(t)\}_{-\infty}^{\infty}$ of (4), where

$$u_n(t) := \begin{cases} 0 & \text{for } \ n \leqslant -1 \\ u_n(x; t) & \text{for } \ 0 \leqslant n \leqslant N \\ 1 & \text{for } \ n \geqslant N+1 \end{cases}$$

and $t \in [0, \tau]$, where τ is defined implicitly by $u_0(x; \tau) = x_1$.

Step 3. Continuation of the solution.

The construction of $T: \bar{X} \to \mathbb{R}^{N+1}$ in Step 2 depended on h. To indicate this dependence we will write from now on T_h instead of T and X_h instead of X. The affinity of h, however, was only needed to show that X_h is convex and nonempty (for suitable d and N).

It turns out that one can still define a continuous map $T_h: \bar{X}_h \to \mathbb{R}^{N+1}$ such that $T_h x \in X$ for all $x \in \bar{X}_h$ where fixed points of T_h correspond to traveling waves, just as we did in Step 2, as long as $h \in \mathscr{B}_{\text{app}}$, where

$$\mathscr{B}_{\text{app}} := \Big\{ h : [0, 1] \to \mathbb{R} : h \text{ is Lipschitz continuous},$$
$$h(0) < 0,\ h(1) > 0,\ h \text{ has a unique zero in } (0, 1),$$
$$\text{and } \int_0^1 h(s)\, ds > 0 \Big\}.$$

Suppose we want to find a traveling wave of (4) for some $h_1 \in \mathscr{B}_{\text{app}}$. In Step 2 we found a traveling wave for the affine function $h_0 \in \mathscr{B}_{\text{app}}$, where

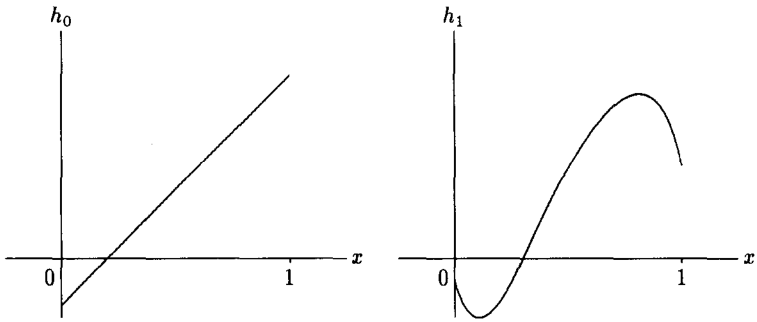

FIG. 4. Deformation of h_0 into h_1 via some homotopy h_λ.

$h_0(x) = x - \frac{1}{4}$. The idea now is to deform continuously h_0 into h_1 (see Fig. 4) and to continue the fixed points of T_{h_0} to the fixed points of T_{h_1}. We will see in Section 5 that this can be done by using a homotopy invariance theorem from index-theory in an appropriate setup.

Step 4. Convergence of the approximate solutions.

Suppose h^k is an approximation of f (see Fig. 5) and $\{u_n^k\}$ is a traveling wave of

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h^k(u_n), \\ u_n &= P(v_n), \qquad n \in \mathbb{Z}. \end{aligned}$$

One expects that as h^k tends to f the corresponding traveling waves $\{u_n^k\}$ approximate a traveling wave of

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \\ u_n &= P(v_n), \qquad n \in \mathbb{Z}. \end{aligned} \tag{7}$$

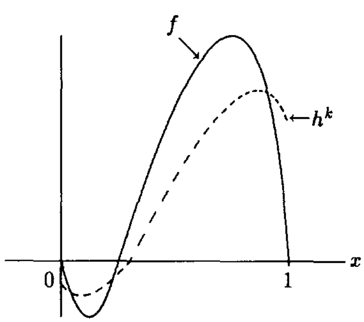

FIG. 5. The function f and an approximation h^k (dotted line) of f.

Let n^k be such that

$$u^k_{n^k}(0) \leqslant \tfrac{1}{2} < u^k_{n^k+1}(0)$$

and call n^k the center of $\{u^k_n\}$. It turns out in Section 6 that $\{u^k_n\}$ shifted by its center, i.e., $\{u^k_{n+n^k}\}$, will indeed converge to a traveling wave of (7). The proof concludes then by showing that a traveling wave of (7) is in fact a strictly monotone traveling wave of the discrete Nagumo equation.

2. Invariance Results

Let $u(t) = \{u_n\}_{n=0}^N$ be the unique solution of the initial value problem (5), where h is any element of $\mathscr{B}_{\text{app}}$. Since $u(t)$ depends continuously on x and h we will also write $u(x; t)$ or $u(x, h; t)$. In this section we will show that certain properties of initial values are invariant under the flow of $u(t)$.

Lemma 1. *Suppose* $x = \{x_n\}_{n=0}^N \in \mathbb{R}^{N+1}$ *satisfies*

(P_1) $0 \leqslant x_0 \leqslant \cdots \leqslant x_N \leqslant 1$,

(P_2) $x_n < x_{n+1}$ *whenever* $0 < x_n < 1$,

(P_3) $d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0$ *whenever* $0 < x_n$, *where* $x_{N+1} := 1$, *and* $x_{-1} := 0$.

Furthermore suppose

$$x_0 = 0, \qquad x_1 = -h(0)/d, \qquad \text{and} \quad d > -h(0).$$

Then $u(t)$ *satisfies* (P_1), (P_2), *and* (P_3) *for all* $t \geqslant 0$.

Proof. Let $t_1 := \sup\{t \geqslant 0 : u(s)$ satisfies (P_1), (P_2), and (P_3) for all $0 \leqslant s \leqslant t\}$. Note that t_1 is well defined because $u(0) = x$ satisfies (P_1), (P_2), and (P_3). One has to show that $t_1 = \infty$.

Observe that $u(t)$ satisfies (P_3) if and only if $\dot{v}_n(t) > 0$ whenever $u_n(t) > 0$. By assumption $x_1 > 0$ and by (P_1) we have $x_n > 0$ for $n = 1, 2, \ldots, N$ and thus by (P_3)

$$\dot{v}_n(0) > 0 \qquad \text{for} \quad n = 1, 2, \ldots, N.$$

Note that

$$\dot{v}_0(0) = dx_1 + h(0) = 0.$$

One checks that $u_0(t)$ is differentiable at $t = 0$ and $\dot{u}_0(0) = 0$. Therefore $h(u_0(t))$ is differentiable at $t = 0$ and $(d/dt)\, h(u_0(t))|_{t=0} = 0$. One calculates then that $\ddot{v}_0(0) > 0$. Now one can conclude that $\dot{v}_0(t) > 0$ for all sufficiently small $t > 0$. Using the continuity of $\dot{v}_n$ and u_n one infers that $t_1 > 0$. Suppose that $t_1 < \infty$. Then either

(i) $u_n(t_1) = u_{n+1}(t_1)$ for some $0 < u_n(t_1) < 1$, or

(ii) $\dot{v}_n(t_1) = 0$ for some $u_n(t_1) > 0$.

We will show that both alternatives are impossible. Suppose (i) occurs. Because $u_{N+1} \equiv 1$, one may choose n such that $u_n(t_1) = u_{n+1}(t_1) < u_{n+2}(t_1)$. It follows that $(d/dt)(v_{n+1} - v_n)|_{t=t_1} = d(u_{n+2}(t_1) - u_{n-1}(t_1)) > 0$, and therefore $u_{n+1}(t_1 - \varepsilon) = v_{n+1}(t_1 - \varepsilon) < v_n(t_1 - \varepsilon) = u_n(t_1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$. This is a contradiction, since $u(t)$ satisfies (P_2) for all $0 \leqslant t \leqslant t_1$.

Consider $u_n(t)$ on the interval $[0, t_1]$. If there exists $t_0 \in (0, t_1)$ such that $u_n(t_0) = 1$, then $u_n(t) = 1$, for all $t \in [t_0, t_1]$, because $v_n(t)$ is nondecreasing on $[0, t_1]$. Therefore the following three cases may occur:

$$\text{(a)}\quad u_n(t) = v_n(t) \qquad \text{for all} \quad t \in [0, t_1],$$

$$\text{(b)}\quad u_n(t) = \begin{cases} v_n(t) & \text{for all} \quad t \in [0, t_0] \\ 1 & \text{for all} \quad t \in (t_0, t_1] \end{cases}$$

$$\text{(c)}\quad u_n(t) = 1 \qquad \text{for all} \quad t \in [0, t_1].$$

In either case the left derivative of $u_n(t)$ exists for every $t \in (0, t_1]$ and is denoted by $\dot{u}_n(t-)$. Suppose now case (ii) occurs. Then $\dot{u}_n(t_1-) = 0$. One calculates that

$$\ddot{v}_n(t_1-) = d(\dot{u}_{n-1}(t_1-) + \dot{u}_{n+1}(t_1-)), \tag{8}$$

where $\dot{u}_{n-1}(t_1-) \geqslant 0$ and $\dot{u}_{n+1}(t_1-) \geqslant 0$. Note that $\ddot{v}_n(t_1-) > 0$ would imply $\dot{v}_n(t_1 - \varepsilon) < 0$ for sufficiently small positive ε, which is impossible. Therefore $\ddot{v}_n(t_1-) = 0$ and by Eq. (8)

$$\dot{u}_{n-1}(t_1-) = 0 \qquad \text{and} \qquad \dot{u}_{n+1}(t_1-) = 0. \tag{9}$$

Note that $v_{n-1}(t_1) > 1$ would give $\dot{v}_n(t_1) = h(1) > 0$ and hence is not possible. Therefore $\dot{v}_{n-1}(t_1) = \dot{u}_{n-1}(t_1-)$ and with (9) $\dot{v}_{n-1}(t_1 = 0$. The arguments which lead to (9) may be repeated with n replaced by $n-1$. By induction we conclude

$$\dot{u}_j(t_1-) = 0 \qquad \text{for} \quad j = 0, 1, ..., n+1. \tag{10}$$

If $v_{n+1}(t_1) > 1$ then $u_{n+1}(t) = 1$ for t sufficiently close to t_1 and therefore by (P_1)

$$\dot{u}_{n+1}(t_1-) = \cdots = \dot{u}_N(t_1-) = 0. \tag{11}$$

On the other hand, if $v_{n+1}(t_1) \leqslant 1$ then $\dot{v}_{n+1}(t_1) = \dot{u}_{n+1}(t_1-) = 0$ and we may repeat the arguments which lead to (9). In either case we will arrive at (11) which together with (10) gives

$$\dot{u}_n(t_1-) = 0 \qquad \text{for} \quad n = 0, 1, ..., N. \tag{12}$$

Let n_1 be the maximal $n \in \{0, 1, ..., N\}$ for which $u_n(t) < 1$ for all $0 \leqslant t < t_1$. Then there exists $t_0 \in [0, t_1)$ such that

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$$

for $n = 0, 1, ..., n_1$, where $u_{n_1+1}(t) = 1$ for $t \in [t_0, t_1]$. By (12) one has

$$\dot{u}_n(t_1) = 0 \qquad \text{for} \quad n = 0, 1, ..., n_1$$

and by the uniqueness of the initial value problem (5),

$$u_n(t) = u_n(t_1) \qquad \text{for} \quad t \in [t_0, t_1], n = 0, 1, ..., n_1.$$

Therefore $\dot{v}_n(t) = 0$ for all $t \in (t_0, t_1)$ and $n \in \{0, 1, ..., n_1\}$ in contradiction to the choice of t_1. ∎

Lemma 1 gives rise to introducing the following sets:

Definition 1. For $d > -h(0)$ and $N \in \mathbb{N}$ define

$$\begin{aligned} C(h, d, N) := \{x \in \mathbb{R}^{N+1} : x_0 = 0, x_1 = -h(0)/d \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1, \\ \text{and } x_n \geqslant n/N, \text{ for } n = 1, 2, ..., N\} \\ O(h, d, N) := \{x \in \mathbb{R}^{N+1} : d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0, \\ \text{for } n = 1, 2, ..., N, \text{ where } x_{N+1} := 1\}. \end{aligned}$$

For shorter notation we will write C, O instead of $C(h, d, N), O(h, d, N)$. Note that C is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$ and O is an open subset of $\mathbb{R}^{N+1}$. The space we are interested in is $\overline{C \cap O}$ for suitable chosen h, d, and N.

Definition 2. Define $t^*: \overline{C \cap O} \to (0, \infty]$ by

$$t^*(x) := \sup\{t : u_0(x; t) < -h(0)/d\}$$

and define

$$T: \{x \in \overline{C \cap O} : t^*(x) < \infty\} \to \mathbb{R}^{N+1}$$

by

$$(Tx)_n := \begin{cases} 0 & \text{for} \quad n=0 \\ u_{n-1}(x; t^*) & \text{for} \quad n=1, ..., N. \end{cases}$$

The rest of this section is concerned with proving that for sufficiently large N

$$T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset C \cap O.$$

LEMMA 2. $T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset O.$

Proof. First consider the case where $x \in C \cap O$ and x satisfies (P_2) of Lemma 1. Then $Tx \in O$ by Lemma 1.

Suppose now $x \in C \cap O$. Let n_1 be the largest $n \in \{0, 1, ..., N\}$ for which $x_n < 1$ and define for $\varepsilon > 0$

$$x_n^\varepsilon := \begin{cases} x_n + n\varepsilon & \text{for} \quad n = 0, ..., n_1 \\ x_n & \text{for} \quad n = n_1 + 1, ..., N. \end{cases}$$

Then for all ε sufficiently small $x^\varepsilon \in C \cap O$ and x^ε satisfies (P_2). Therefore $u(x^\varepsilon, t^*(x))$ satisfies (P_3) by Lemma 1. Since $x^\varepsilon \to x$ as $\varepsilon \to 0$ and u depends continuously on its initial conditions, one concludes that $y := u(x; t^*)$ satisfies $d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) \geqslant 0$ whenever $y_n > 0$. By the same arguments as in the proof of Lemma 1(ii), we can lead $d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) = 0$ to a contradiction. It follows that $u(x; t^*)$ satisfies (P_3) and therefore $Tx \in O$.

The general case where $x \in \overline{C \cap O}$ follows also by approximation. Indeed, there exists a sequence $\{x^k\}$ in $C \cap O$ such that $x^k \to x$ as $k \to \infty$. Since $u(x^k, t^*(x))$ satisfies (P_3) for each k we may argue as above to conclude that $u(x, t^*(x))$ satisfies (P_3) and therefore $Tx \in O$. ∎

LEMMA 3. *For every $h \in \mathscr{B}_{\text{app}}$ there exists a positive δ such that*

$$T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset C$$

for $N \geqslant 1/\delta$.

Proof. Let $y := Tx$. By the definition of t^* one has $y_0 = 0$ and $y_1 = -h(0)/d$. By the same argument as in Lemma 2 one shows that $y_1 \leqslant y_2 \cdots \leqslant y_N \leqslant 1$. Therefore, it remains to show the existence of a number $\delta > 0$ such that $y_n \geqslant n/N$ for $n = 1, ..., N$, whenever $N \geqslant 1/\delta$. We will construct δ explicitly. Let

$$\delta_1 := -h(0)/d,$$

$$\delta_2 := \max\{x - a : a < x \leqslant 1 \text{ and } b(x) \leqslant (d/2)\, \delta_1\},$$

where $b(x) := \max_{a \leqslant s \leqslant x} h(s)$, $x \in [a, 1]$, and $a \in (0, 1)$ is the unique zero of h,

$$\delta_3 := \min\{\tfrac{1}{2}\delta_1, \delta_2\},$$

$$\delta_4 := \delta_3/2,$$

$$\delta_5 := \min\{h(s)/4d : a + \delta_3/2 \leqslant s \leqslant 1\},$$

and

$$\delta_6 := \min\{d\delta_5/m_2, -h(0)/dm_1\}\, d\delta_5,$$

where $m_1 := d + \sup_{0 \leqslant s \leqslant 1} |h(s)|$, and $m_2 := m_1(4d + \sup_{s \neq t} |(h(s) - h(t))/(s-t)|)$. Finally, let

$$\delta := \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}. \tag{13}$$

Suppose that $N \geqslant 1/\delta$. We will show that

$$y_n \geqslant n/N \qquad \text{for} \quad n = 1, 2, ..., N.$$

Suppose $y_n \leqslant a$, where a denotes the unique zero of h. Then

$$y_{n+1} - y_n > y_n - y_{n-1} - h(y_n)/d \geqslant y_n - y_{n-1},$$

because $y \in O(h, d, N)$ and $h(x) \leqslant 0$ for $x \leqslant a$. We conclude by induction that

$$y_{i+1} - y_i > y_1 - y_0 = -h(0)/d = \delta_1,$$

i.e.,

$$y_i \geqslant i\delta \geqslant i/N \qquad \text{for} \quad i = 1, ..., n+1.$$

If $y_{N-1} \leqslant a$ we are done. Otherwise there is an index n_0 such that

$$y_{n0-1} \leqslant a \qquad \text{and} \qquad y_{n_0} > a.$$

It remains to check that

$$y_n \geqslant n/N \qquad \text{for} \quad n = n_0 + 1, ..., N.$$

Either $y_{n_0} \leqslant a + \delta_2$ in which case

$$y_{n_0+1} - y_{n_0} > (y_{n_0} - y_{n_0-1}) - (1/d)\, h(y_{n_0}) \geqslant \delta_1 - \tfrac{1}{2}\delta_1 = \tfrac{1}{2}\delta_1,$$

or $y_{n_0} > a + \delta_2$. In either case, it suffices to show that

$$y_n \geqslant n/N \qquad \text{for} \quad y_n \geqslant a + \delta_3.$$

Suppose the contrary and let n_1 be the smallest index for which

$$y_{n_1} \geqslant a + \delta_3 \qquad \text{and} \qquad y_{n_1} < n_1/N.$$

Then

$$(n_1 - 2)/N \leqslant x_{n_1 - 2} \leqslant x_{n_1 - 1} \leqslant y_{n_1} < n_1/N$$

and

$$x_{n_1 - 1} \geqslant a + \delta_3/2.$$

Since

$$1/N \leqslant \delta_5 \leqslant h(x_{n_1 - 1})/4d,$$

it follows that

$$x_{n_1 - 1} - x_{n_1 - 2} < \frac{n_1}{N} - \frac{n_1 - 2}{N} = \frac{2}{N} \leqslant \frac{h(x_{n_1 - 1})}{2d}. \tag{14}$$

Suppose $0 < u_n(0) < 1$ and $0 < u_n(t) < 1$. Then using

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$$

one estimates that

$$|\dot{u}_n(t) - \dot{u}_n(0)| \leqslant tm_2.$$

This implies

$$\dot{u}_n(t) \geqslant \dot{u}_n(0) - tm_2$$

and thus

$$\dot{u}_n(t) \geqslant \tfrac{1}{2}\dot{u}_n(0) \qquad \text{for} \quad 0 \leqslant t \leqslant \dot{u}_n(0)/2m_2.$$

This gives

$$u_n(t) - u_n(0) = \int_0^t \dot{u}_n(s)\, ds \geqslant t\,\frac{1}{2}\,\dot{u}_n(0) \tag{15}$$

for $0 \leqslant t \leqslant \dot{u}_n(0)/2m_2$. One estimates

$$\begin{aligned}
\dot{u}_{n_1 - 1}(0) &= d(u_{n_1 - 2}(0) - 2u_{n_1 - 1}(0) + u_{n_1}(0)) + h(u_{n_1 - 1}(0)) \\
&\geqslant h(u_{n_1 - 1}(0)) - d(u_{n_1 - 1}(0) - u_{n_1 - 2}(0)) \\
&= h(x_{n_1 - 1}) - d(x_{n_1 - 1} - x_{n_1 - 2}) \\
&\geqslant h(x_{n_1 - 1})/2, \qquad \text{by (14).}
\end{aligned}$$

Using this together with (15) for $n = n_1 - 1$ we obtain

$$u_{n_1-1}(t) - x_{n_1-1} \geqslant t\,\frac{h(x_{n_1-1})}{4} \qquad \text{for} \quad 0 \leqslant t \leqslant \frac{h(x_{n_1-1})}{4m_2}. \tag{16}$$

Using

$$-\frac{h(0)}{d} = u_0(t^*) - u_0(0) = \int_0^{t^*} \dot{u}_0(s)\,ds \leqslant t^* \sup_s \dot{u}_0(s)$$

and $\dot{u}_0 = d(u_1 - 2u_0) + h(u_0)$, one estimates that $t^* \geqslant -h(0)/dm_1$. We infer now from (16) that

$$u_{n_1-1}(t^*) - x_{n_1-1} \geqslant \delta_6$$

and thus

$$y_{n_1} \geqslant \delta_6 + x_{n_1-1} \geqslant n_1/N.$$

This is a contradiction to $y_{n_1} < n_1/N$. ∎

3. A Priori Estimates and Properties

In this section we will prove a priori estimates and properties of fixed points of T, where T is defined in Definition 2. Recall from the introduction, that fixed points of T correspond to traveling waves of (4). We will denote by $a(h)$ or a the unique zero of $h \in \mathscr{B}_{\text{app}}$ in (0, 1).

Lemma 4. *Suppose x is a fixed point of T, and let $\tau := t^*(x)$. Let*

$$e(h) := \min_{a/4 \leqslant s \leqslant a/2} \frac{-h(s)}{d}, \qquad m(h) := \min\left\{\frac{a(h)}{4}, e(h)\right\},$$

and

$$M(h) := \max_{0 \leqslant s \leqslant 1} \{2d + h(s)\}.$$

Then $\tau \geqslant \tau_0(h) := m(h)/M(h) > 0$.

Proof. There are two cases. First suppose that there exists an integer n such that $a/4 \leqslant x_n \leqslant a/2$. Since $x = Tx \in O$ by Lemma 2, we have $d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0$, and thus $x_{n+1} - x_n > -h(x_n)/d + x_n - x_{n-1}$. This implies $x_{n+1} - x_n \geqslant e(h)$.

If there is no index n such that $a/4 \leqslant x_n \leqslant a/2$, then there exists n such that $x_{n+1} - x_n > a/4$. In either case there exists an integer n such that

$$x_n \leqslant a/2, \qquad \text{and} \qquad x_{n+1} - x_n \geqslant m(h). \tag{17}$$

Since $\dot{u}_n(t) \leqslant M(h)$ for all $t \geqslant 0$ we obtain

$$m(h) \leqslant u_{n+1}(0) - u_n(0) = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(t)\, dt \leqslant \tau M(h),$$

i.e.,

$$\tau \geqslant m(h)/M(h). \quad \blacksquare$$

LEMMA 5. *Let*

$$\varepsilon_0 := \tfrac{1}{2}\min\{a, 1-a\}$$

and for $x = \{x_n\}_0^N$ *let* $\#(x, \varepsilon)$ *be the number of* x_n*'s in* x *such that* $\varepsilon < x_n < 1 - \varepsilon$. *Then for all* $\varepsilon \in [0, \varepsilon_0]$ *there exists a bound* $S(\varepsilon, h)$ *independent of* N *such that*

$$\#(x, \varepsilon) \leqslant S(\varepsilon, h)$$

for all $x \in \{x \in \bigcup_N \overline{C \cap O} : Tx = x\}$.

Proof. We pick any $h \in \mathscr{B}_{\text{app}}$, $\varepsilon \in [0, \varepsilon_0]$ and suppose x is a fixed point of T. We construct $S(\varepsilon, h)$ in four steps.

Step 1. Let

$$p(\varepsilon, h) := \max_{\varepsilon \leqslant s \leqslant a/2} \frac{d}{-h(s)}$$

and suppose

$$\varepsilon < x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_j \leqslant a/2.$$

Then

$$\begin{aligned} 1 \geqslant x_{j+1} - x_i = \sum_{n=i}^{j} (x_{n+1} - x_n) &\geqslant \sum_{n=i}^{j} (x_n - x_{n-1}) - \frac{1}{d} \sum_{n=i}^{j} h(x_n) \\ &\geqslant -\frac{1}{d} \sum_{n=i}^{j} h(x_n) \geqslant \frac{j+1-i}{p(\varepsilon, h)}. \end{aligned}$$

Hence

$$j + 1 - i \leqslant p(\varepsilon, h),$$

i.e., the number of x_n's with $\varepsilon < x_n \leqslant a/2$ is bounded above by $p(\varepsilon, h)$.

Step 2. In the proof of Lemma 4 (cf. (17)) it is shown that there exists an integer n_0 such that

$$x_{n_0} \leqslant a/2, \qquad \text{and} \qquad x_{n_0+1} - x_{n_0} \geqslant m(h),$$

where $m(h)$ is defined in Lemma 4. Since

$$x_{n+1} - x_n > x_n - x_{n-1} - (1/d)\, h(x_n) \geqslant x_n - x_{n-1}$$

for all $0 \leqslant x_n \leqslant a$, we obtain

$$x_{n+1} - x_n \geqslant m(h) \qquad \text{for all} \quad a/2 \leqslant x_n \leqslant a.$$

We conclude that the number of x_n's with $a/2 \leqslant x_n \leqslant a$ is bounded above by $1 + a/2m(h)$.

Step 3. Let

$$b(x, h) := \max_{a \leqslant s \leqslant x} \frac{h(s)}{d},$$

$$\sigma_1(h) := \max\{x - a : a \leqslant x \leqslant 1, b(x, h) \leqslant m(h)/2\},$$

and let

$$\sigma_2(h) := \min\{\sigma_1(h), m(h)/2\}.$$

Note that $\sigma_2(h) > 0$. There is an integer n_0, such that

$$x_{n_0-1} \leqslant a \qquad \text{and} \qquad x_{n_0} > a.$$

Either $x_{n_0} \leqslant a + \sigma_1(h)$, in which case

$$x_{n_0+1} - x_{n_0} > (x_{n_0} - x_{n_0-1}) - (1/d)\, h(x_{n_0}) \geqslant m(h) - \tfrac{1}{2} m(h) = m(h)/2$$

or $x_{n_0} > a + \sigma_1(h)$. In both cases there is at most one x_n such that $a < x_n \leqslant a + \sigma_2(h)$.

Step 4. Suppose $a + \sigma_2(h) \leqslant x_n \leqslant x_{n+1} < 1 - \varepsilon$. Then for $\tau := t^*(x)$

$$x_{n+1} - x_{n-1} \geqslant x_{n+1} - x_n = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(s)\, ds. \tag{18}$$

For $0 \leqslant s \leqslant \tau$ we estimate $\dot{u}_n(s)$ as follows:

$$\begin{aligned} \dot{u}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \\ &\geqslant h(u_n) - d(u_n - u_{n-1}) \\ &\geqslant \min_{x_n \leqslant s \leqslant x_{n+1}} h(s) - d(x_{n+1} - x_{n-1}). \end{aligned}$$

Using this estimate in (18), one obtains

$$x_{n+1}-x_{n-1}\geqslant\frac{\tau}{d\tau+1}\big(\min_{x_n\leqslant s\leqslant x_{n+1}} h(s)\big).$$

Since

$$\frac{\tau}{d\tau+1}\geqslant\frac{\tau_0}{d\tau_0+1},\qquad\text{for}\quad\tau\geqslant\tau_0,$$

one concludes by Lemma 4 that

$$x_{n+1}-x_{n-1}\geqslant\sigma_3:=\frac{\tau_0(h)}{d\tau_0(h)+1}\big(\min_{a+\sigma_2\leqslant s\leqslant 1-\varepsilon} h(s)\big).$$

Therefore, if one assumes that

$$a+\sigma_2\leqslant x_i\leqslant\cdots\leqslant x_{j+1}<1-\varepsilon,$$

then

$$2\geqslant\sum_{n=i}^{j}(x_{n+1}-x_{n-1})\geqslant(j+1-i)\,\sigma_3(h).$$

This shows that the number of x_n's such that

$$a+\sigma_2\leqslant x_n<1-\varepsilon,$$

is bounded above by $2/\sigma_3(h)+1$. Summarizing the results in Steps 1, 2, 3, and 4, we infer

$$\#(x,\varepsilon)\leqslant p(\varepsilon,h)+\left(1+\frac{a(h)}{2m(h)}\right)+1+\left(\frac{2}{\sigma_3(h)}+1\right):=S(\varepsilon,h).\quad\blacksquare$$

In Definition 2 we defined T for all $x\in\overline{C(h,d,N)\cap O(h,d,N)}$ for which $t^*(x)<\infty$. The following two lemmas will be used to show that for d sufficiently large $t^*(x)<\infty$ for all $x\in\overline{C\cap O}$.

LEMMA 6. *Let* $x\in C(h,d,N)\cap O(h,d,N)$, $\dot{u}(t)=\dot{u}(x,t)$, *and* $D>0$. *Suppose that for all* $n\in\{1,2,...,N\}$, $0<u_n(t)<1$ *implies* $\dot{u}_n(t)<D$. *Then for all* $k\in\mathbb{N}$ *for which* $d\geqslant k^2(D+\sup|h|)$, $u_n(t)-u_{n-1}(t)\leqslant 2/k$ *for* $n=1,2,...,N$.

Proof. For a shorter notation let

$$\Delta_n:=u_n(t)-u_{n-1}(t).$$

For $0 < u_n(t) < 1$ one has

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$$

and therefore

$$\Delta_{n+1} - \Delta_n = (1/d)(\dot{u}_n(t) - h(u_n(t))).$$

Using the assumption $\dot{u}_n(t) < D$ and $d \geqslant k^2(D + \sup|h|)$ one obtains

$$|\Delta_{n+1} - \Delta_n| \leqslant 1/k^2 \qquad \text{if} \quad 0 < u_n < 1.$$

If $u_n(t) = 0$, then $n = 0$, $t = 0$, and $u_1(0) = -h(0)/d$. Therefore,

$$|\Delta_{n+1} - \Delta_n| = -h(0)/d < 1/k^2.$$

If $u_n(t) = 1$, then $\Delta_{n+1} = 0$ and one calculates that

$$|\Delta_{n+1} - \Delta_n| = \Delta_n < h(1)/d < 1/k^2$$

by using $d(\Delta_{n+1} - \Delta_n) + h(u_n) > 0$. In either case one has

$$|\Delta_{n+1} - \Delta_n| \leqslant 1/k^2.$$

Now assume contrary to the conclusion that there exists n_0 such that

$$\Delta_{n_0} > 2/k.$$

Then

$$\Delta_{n_0+m} = \Delta_{n_0} - \sum_{i=1}^{m} (\Delta_{n_0+i-1} - \Delta_{n_0+i}) > 2/k - m/k^2$$

and thus

$$\Delta_{n_0+m} > 1/k \qquad \text{for} \quad m = 0, 1, ..., k.$$

This implies that $n_0 + k \leqslant N$ because $\Delta_{N+1} = 0$. Then

$$1 \geqslant u_{n_0+k}(t) - u_{n_0-1}(t) = \sum_{m=0}^{k} \Delta_{n_0+m} > \frac{k+1}{k},$$

which is a contradiction. ∎

LEMMA 7. *There exists a number* d_1 *which depends only on* $\sup|h|$, $\sup_{s \neq t} |(h(s) - h(t))/(s-t)|$, *and* $\int_0^1 h(s)\,ds$, *such that for all* $x \in C(h, d, N) \cap O(h, d, N)$, $t \in [0, t^*)$, $d > d_1$, *the following holds*:

$$\sup_n \dot{u}_n(t) \geqslant \frac{1}{2} \int_0^1 h(s)\,ds.$$

The sup *here is taken over all* n *for which* $0 < u_n < 1$.

Proof. Let $n_1 > 0$ be such that $u_{n_1}(t) < 1$. In the following t is fixed and we write u_n instead of $u_n(t)$. Then

$$d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) = \dot{u}_n \qquad \text{for} \quad n = 0, ..., n_1.$$

This implies

$$h(u_n)(u_{n+1} - u_n) + d(u_{n-1}u_{n+1} - 2u_n u_{n+1} + u_{n+1}^2 - u_{n-1}u_n + 2u_n^2 - u_n u_{n+1}) = \dot{u}_n(u_{n+1} - u_n).$$

and

$$h(u_n)(u_n - u_{n-1}) + d(u_{n-1}u_n - 2u_n^2 + u_n u_{n+1} - u_{n-1}^2 + 2u_{n-1}u_n - u_{n-1}u_{n+1}) = \dot{u}_n(u_n - u_{n-1}).$$

Adding the two equations gives

$$h(u_n)(u_{n+1} - u_n) + h(u_n)(u_n - u_{n-1}) + d(u_{n+1}^2 - u_{n-1}^2 + 2u_{n-1}u_n - 2u_n u_{n+1}) = \dot{u}_n(u_{n+1} - u_{n-1}) \leqslant (\max_{0 \leqslant n \leqslant n_1} \dot{u}_n)(u_{n+1} - u_{n-1}).$$

Adding over n from 0 to n_1 gives

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) + d((u_{n_1+1} - u_{n_1})^2 - u_0^2) \leqslant (\max_{0 \leqslant n \leqslant n_1} \dot{u}_n)(u_{n+1} + u_{n_1} - u_0)$$

and therefore

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \leqslant du_0^2 + 2 \max_{0 \leqslant n \leqslant n_1} \dot{u}_n.$$

Recall that $u_0 \leqslant -h(0)/d$ and thus

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \leqslant h(0)^2/d + 2 \max_{0 \leqslant n \leqslant n_1} \dot{u}_n. \tag{19}$$

Assume now the contrary of the claim, i.e., suppose

$$\dot{u}_n < D := \frac{1}{2} \int_0^1 h(s)\, ds \qquad \text{for} \quad n = 0, 1, ..., n_1.$$

Denote by $n_1(d)$ the largest $n \in \{0, ..., N\}$ such that $u_n < 1$. Then there exists a number d_1 by Lemma 6 such that

$$d_1 > \frac{2h(0)^2}{\int_0^1 h(s)\,ds},$$

$$\left|\sum_{n=0}^{n_1(d)} h(u_n)(u_{n+1} - u_n) - \int_0^1 h(s)\,ds\right| < \frac{1}{4}\int_0^1 h(s)\,ds,$$

and

$$\left|\sum_{n=0}^{n_1(d)} h(u_n)(u_n - u_{n-1}) - \int_0^1 h(s)\,ds\right| < \frac{1}{4}\int_0^1 h(s)\,ds$$

for $d > d_1$. Note that d_1 depends only on

$$\sup |h|, \qquad \sup_{s \neq t}\left|\frac{h(s) - h(t)}{s - t}\right|, \qquad \text{and} \qquad \int_0^1 h(s)\,ds.$$

This gives a contradiction in (19) for $d > d_1$. ∎

4. Solution of the Simplified Problem

Let $d_2 := \max\{8, d_1\}$, where d_1 is chosen according to Lemma 7. In this section it will be shown that the simplified problem has a solution for $d > d_2$. Let δ be defined by Eq. (13) so that the conclusion of Lemma 3 holds. For the rest of this section d is any number larger than d_2 and N an integer larger than $1/\delta$. For shorter notation let

$$C_0 := C(h_0, d, N) \qquad \text{and} \qquad O_0 := O(h_0, d, N),$$

where $h_0(x) = x - \frac{1}{4}$. All u_n's are nondecreasing by Lemma 1 and the proof of Lemma 2. Therefore one concludes from Lemma 7 that for $t \in (0, t^*)$

$$N \geq \sum_{n=0}^{N} u_n(t) - \sum_{n=0}^{N} u_n(0) = \int_0^t \sum_{n=0}^{N} \dot{u}_n(s-)\,ds \geq t\,\frac{1}{2}\int_0^1 h_0(s)\,ds$$

and thus

$$t^*(x) \leq 2N \Big/ \int_0^1 h_0(s)\,ds =: M^*. \tag{20}$$

Denote by T_0 the map $T = T(h_0)$ which has been defined in Definition 2. Since $t^*(x) \leq M^*$, the map T_0 is defined on $\overline{C_0 \cap O_0}$. The map T_0 has a fixed point by Brouwer's fixed point theorem if the following holds:

(i) $\overline{C_0 \cap O_0}$ is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$.

(ii) $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$.

(iii) T_0 is continuous.

(iv) $C_0 \cap O_0$ is nonempty.

The proof of (i) is straightforward and (ii) follows from Lemmas 2 and 3. In order to show that T_0 is continuous define

$$r: \overline{C_0 \cap O_0} \times [0, M^*] \to \mathbb{R}$$

by $r(x, t) := u_0(x; t)$, where M^* is defined in (20) and define

$$s: \overline{C_0 \cap O_0} \times [0, M^*] \to \mathbb{R}$$

by $s(x, t) := -h(0)/d$. Then r and s are continuous functions. Let

$$G(t^*) := \{(x, t) : x \in \overline{C_0 \cap O_0},\ t = t^*(x)\}$$

be the graph of t^*. Then

$$G(t^*) = \{(x, t) : r(x, t) = s(x, t)\}$$

is closed in $\overline{C_0 \cap O_0} \times [0, M^*]$ and therefore compact. Since the graph of t^* is compact, it follows that t^* is continuous. Inspection of the definition of T_0 shows then that T_0 is continuous. We now show (iv) by constructing an $x \in C_0 \cap O_0$. In this construction let $a = \frac{1}{4}$ and thus $h_0(x) = x - a$.

Step 1. Let $x_0 := 0$ and define inductively

$$x_{n+1} := x_n + (n+1)\, a/d \qquad \text{for} \quad n = 0, 1, ..., n_0,$$

where n_0 is determined by the condition

$$x_{n_0} \leqslant a \qquad \text{and} \qquad x_{n_0+1} > a.$$

Then for $n = 1, ..., n_0$ one has

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n > 0.$$

Step 2. Since

$$x_{n_0} \leqslant a < x_{n_0+1} \Leftrightarrow \frac{n_0(n_0+1)\, a}{2d} \leqslant a < \frac{(n_0+1)(n_0+2)\, a}{2d},$$

it follows that

$$(n_0+1)\, a/d \leqslant 2a/n_0 \qquad \text{and} \qquad 2d < (n_0+1)(n_0+2),$$

which implies

$$n_0 > \sqrt{2d} - 2 > 2.$$

Here we used that $d > 8$. Hence

$$x_{n_0+1} = x_{n_0} + (n_0 + 1)\, a/d \leqslant a + 2a/n_0 < 2a$$

which proves $x_{n_0+1} < \frac{1}{2}$. Let

$$x_{n+1} := x_n + (n_0 + 1)\, a/d \qquad \text{for} \quad n = n_0 + 1, ..., n_1,$$

where n_1 is determined by the condition

$$x_{n_1} \leqslant \tfrac{1}{2} \qquad \text{and} \qquad x_{n_1+1} > \tfrac{1}{2}.$$

It could be that $n_0 = n_1$ in which case no new x_n's are constructed. So suppose that $n_1 \geqslant n_0 + 1$. Then for $n = n_0 + 1, ..., n_1$ one calculates that

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0.$$

Step 3. Next let

$$x_{n+1} := x_n + (n_1 + n_0 + 1 - n)\, a/d \qquad \text{for} \quad n = n_1 + 1, ..., n_0 + n_1.$$

Then for $n = n_1 + 1, ..., n_0 + n_1$ one calculates that

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - 2a > 0.$$

Since

$$x_{n_0+n_1+1} = x_{n_1+1} + \left(\sum_{n=1}^{n_0} n \right) \frac{a}{d} = x_{n_1+1} + x_{n_0}$$

and

$$x_{n_1+1} = x_{n_1} + (n_0 + 1)\, a/d \leqslant \tfrac{1}{2} + x_{n_0+1} - x_{n_0} < 1 - x_{n_0},$$

one concludes

$$x_{n_0+n_1+1} < 1.$$

Step 4. Define

$$x_{n+1} := x_n + a/d \qquad \text{for} \quad n = n_0 + n_1 + 1, ..., n_2,$$

where n_2 is determined by the condition

$$1 - a/d \leqslant x_{n_2+1} < 1.$$

Then for $n = n_0 + n_1 + 1, ..., n_2$ we obtain

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0.$$

Step 5. Finally let

$$x_n := 1 \qquad \text{for all} \quad n = n_2 + 2, ..., N.$$

Then for $n = n_2 + 1, ..., N$ we have

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) > x_n - 2a > 0. \quad \blacksquare$$

5. Solution of the Approximate Problem

Let $h_1 \in \mathscr{B}_{\text{app}}$. We will now construct a homotopy h_λ that deforms h_0 continuously into h_1 in the set $\mathscr{B}_{\text{app}}$. Denote by a_0 the unique zero of h_0 and by a_1 the unique zero of h_1. Let

$$h(\lambda, x) := \begin{cases} \left(1 - \dfrac{\lambda}{a_0}\right) h_0(x) + \dfrac{\lambda}{a_0}\, g(a_0, x) & \text{for} \quad 0 \leqslant \lambda \leqslant a_0 \\ g(\lambda, x) & \text{for} \quad a_0 \leqslant \lambda \leqslant a_1 \\ \left(\dfrac{1-\lambda}{1-a_1}\right) g(a_1, x) + \dfrac{\lambda - a_1}{1 - a_1}\, h_1(x) & \text{for} \quad a_1 \leqslant \lambda \leqslant 1, \end{cases}$$

where $g(\lambda, x)$: $[a_0, a_1] \times [0, 1] \to \mathbb{R}$ is defined by

$$g(\lambda, x) := \max\left\{-\frac{(1-\lambda)^2}{4\lambda}, x - \lambda\right\}.$$

A calculation shows that

$$\int_0^1 g(\lambda, x)\, dx = \frac{(1-\lambda)^2}{4} + \frac{(1-\lambda)^4}{32\lambda^2} > 0 \qquad \text{for} \quad a_0 \leqslant \lambda \leqslant a_1.$$

Then $h_\lambda := h(\lambda, \cdot)$ is a homotopy that deforms $h_0(x)$ continuously into $h_1(x)$ in the set $\mathscr{B}_{\text{app}}$. Let

$$C_\lambda := C(h_\lambda, d, N), \qquad O_\lambda := (h_\lambda, d, N),$$
$$\mathscr{C} := \{(\lambda, x) : \lambda \in [0, 1], x \in C_\lambda\},$$

and

$$\mathscr{O} := \{(\lambda, x) : \lambda \in [0, 1], x \in O_\lambda\}.$$

By Lemma 7 there exists a constant $d_3 > d_2$ such that

$$\sup_n \dot{u}_n(t) \geqslant \frac{1}{2} \int_0^1 h_\lambda(s)\, ds \qquad \text{for all} \quad (\lambda, x) \in \overline{\mathscr{C} \cap \mathcal{O}}.$$

We will now specify the parameters d and N. Let d be any number greater than d_3. An inspection of the definition of δ in (13) shows that δ depends continuously on h. One can choose therefore a δ such that the conclusion of Lemma 2 holds for all h_λ. Then let N be some integer greater than $1/\delta$. Next, we define

$$T: \overline{\mathscr{C} \cap \mathcal{O}} \to \mathbb{R}^{N+1}$$

similarly to Definition 2. First let

$$t^*: \overline{\mathscr{C} \cap \mathcal{O}} \to (0, \infty)$$

be defined by

$$t^*(\lambda, x) := \sup\{t : u_0(x, h_\lambda; t) < -h_\lambda(0)/d\}.$$

Analogous to (20) one has

$$t^*(\lambda, x) \leqslant 2N \Big/ \left(\min_{0 \leqslant \lambda \leqslant 1} \int_0^1 h_\lambda(s)\, ds \right) =: M^*.$$

One can show that t^* is continuous just as we did in Section 4. Let $T(\lambda, x)$ be defined by

$$(T(\lambda, x))_n := \begin{cases} 0 & \text{for} \quad n = 0 \\ u_{n-1}(x, h_\lambda; t^*(\lambda, x)) & \text{for} \quad n = 1, ..., N. \end{cases}$$

Then T is continuous. We will use the following notation. If $X \subset [0, 1] \times \mathbb{R}^{N+1}$ then let

$$X_\lambda := \{x : (\lambda, x) \in X\}$$

be the "slice" of X at λ. Furthermore we denote by $F_\lambda : X_\lambda \to \mathbb{R}^{N+1}$ the restriction of $F: X \to \mathbb{R}^{N+1}$ to the slice X_λ. We will use the following general homotopy invariance theorem from index-theory (cf. [1]).

THEOREM 2. *Let Λ be a nonempty compact interval, let A be a retract of some Banach space, and let U be an open subset of $\Lambda \times A$. Suppose $F: \bar{U} \to A$ is a compact map such that $F(\lambda, x) \neq x$ for every $(\lambda, x) \in \partial U$. Then $i(F_\lambda, U_\lambda, A)$ is well defined and independent of $\lambda \in \Lambda$.*

In order to apply this theorem let

$$\Phi\colon [0,1]\times\mathbb{R}^{N+1}\to[0,1]\times\mathbb{R}^{N+1}$$

be defined by $\Phi(\lambda, x)=(\lambda, y)$, where $y=\{y_n\}_{n=0}^N$ is given by

$$y_n := \begin{cases} \dfrac{h_\lambda(0)}{h_0(0)}\, x_n & \text{for} \quad n=1 \\ x_n & \text{for} \quad n\neq 1. \end{cases}$$

Then Φ is an homeomorphism. Let $U := \Phi^{-1}(\mathscr{C}\cap\mathcal{O})$ and $A := C_0$. Note that A is a retract of $\mathbb{R}^{N+1}$, because C_0 is closed and convex. Define

$$F\colon \bar{U}\to A \text{ by } F(\lambda, x) := \Phi_\lambda^{-1}(T\circ\Phi(\lambda, x)).$$

It follows from Lemmas 2 and 3 that

$$T(\lambda, x)\in C_\lambda\cap O_\lambda \qquad \text{for} \quad (\lambda, x)\in\mathscr{C}\cap\mathcal{O}.$$

In fact, by the same arguments as in the proof of Lemma 1(ii), it follows that

$$T(\lambda, x)\in C_\lambda\cap O_\lambda \qquad \text{for} \quad (\lambda, x)\in\overline{\mathscr{C}\cap\mathcal{O}}.$$

Therefore $F(\bar{U})\subset A$. Since $\mathcal{O}$ is an open set, $U=([0,1]\times A)\cap\Phi^{-1}(\mathcal{O})$ is an open subset of $[0,1]\times A$ (in the relative topology). Now one can apply Theorem 2 and conclude that $i(F_\lambda, U_\lambda, A)$ is well defined and independent of $\lambda\in[0,1]$. Since $\Phi_0(x)\equiv x$ it follows that $F_0\equiv T_0$ and $U_0=C_0\cap O_0$. Therefore U_0 is convex and one calculates that

$$i(F_0, U_0, A)=1$$

via the homotopy $(1-\tau)\,x^0+\tau F_0$, where $x^0\in U_0$. Therefore

$$i(F_1, U_1, A)=1$$

and by the solution property of the index, there exists an $x\in U_1$ such that $F_1 x=x$. Therefore $\Phi_1(x)$ is a fixed point of T_1. As in Section 4 the fixed point of T_1 corresponds to a traveling wave solution.

6. Convergence of Approximate Solutions

Let $\{h_k\}$ be a sequence in $\mathscr{B}_{\text{app}}$ that converges to f in the norm defined by

$$\|h\| := \sup|h| + \sup_{s\neq t}\left|\frac{h(s)-h(t)}{s-t}\right|.$$

We have seen in Section 5 that for every $h_k \in \mathscr{B}_{\text{app}}$ there exists a number $d_3(h_k)$ such that

$$\begin{aligned} \dot{v}_n &= d(u_{n_1} - 2u_n + u_{n+1}) + h_k(u_n), \\ u_n &= P(v_n), \qquad n \in \mathbb{Z} \end{aligned} \tag{21}$$

has a traveling wave solution for $d > d_3(h_k)$. Recall that d_3 was defined according to Lemma 7 and depended only on

$$\sup |h_k|, \qquad \sup_{s \neq t} \left| \frac{h_k(s) - h_k()t)}{s-t} \right|, \qquad \text{and} \qquad \int_0^1 h_k(s)\, ds.$$

Therefore there exists a number $d^* < \infty$ such that (21) has a traveling wave solution for all $d > d^*$. Let d be any number greater than d^*. Then for every $k \in \mathbb{N}$ there exists a traveling wave solution, say u^k, v^k, of (21). Let $x^k = \{x_n^k\}_{n=-\infty}^{\infty} \in l^\infty$ be defined by $x_n^k := v_n^k(0)$. Then there exists an integer n_k such that

$$x_{n_k}^k \leqslant \tfrac{1}{2} < x_{n_k+1}^k.$$

Let $y^k = \{y_n^k\}_{n=-\infty}^{\infty}$ be defined by

$$y_n^k := x_{n+n_k}^k \qquad \text{for} \quad n \in \mathbb{Z}.$$

In other words the sequence x^k is shifted to the sequence y^k in order that

$$y_0^k \leqslant \tfrac{1}{2} < y_1^k.$$

An inspection of $S(\varepsilon, h)$ in Lemma 5, cf. (3), shows that $\limsup_{k \to \infty} S(\varepsilon, h^k) < \infty$ for all $\varepsilon > 0$. Therefore the sequence $\{y^k\}$ has a convergent subsequence in l^∞, i.e., we may assume without loss of generality that

$$\lim_{k \to \infty} y^k = y \qquad \text{for some} \quad y \in l^\infty \text{ with } \lim_{n \to -\infty} y_n = 0 \text{ and } \lim_{n \to \infty} y_n = 1. \tag{22}$$

It follows from Lemmas 4 and 7 that

$$0 < \inf_k \tau_0(h_k) \leqslant \tau_k \leqslant \sup_k \frac{2}{\int_0^1 h_k(s)\, ds} < \infty, \qquad \text{where} \quad \tau_k := t^*(x^k).$$

The first inequality follows from Lemma 4 and for the third note that

$$\begin{aligned} 1 = \sum_n (u_{n+1}(0) - u_n(0)) &= \sum_n (u_n(\tau) - u_n(0)) \\ &= \sum_n \int_0^\tau \dot{u}_n(s)\, ds \\ &= \int_0^\tau \sum_n \dot{u}_n(s)\, ds \\ &\geqslant \tau \inf_{0 \leqslant t \leqslant \tau} (\sup_n \dot{u}_n(t)). \end{aligned}$$

Therefore we may assume without loss of generality that

$$\lim_{k \to \infty} \tau_k = \tau \qquad \text{for some} \quad 0 < \tau < \infty. \tag{23}$$

We will show now that the solution $\{u_n\}$ of the initial value problem

$$\begin{aligned} \dot{u}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \\ v_n(0) &= y_n, \qquad n \in \mathbb{Z}. \end{aligned}$$

is a traveling wave with velocity $c = 1/\tau$, where τ is determined in (23) and y is determined in (22).

First note that $0 \leqslant u_n(t) \leqslant 1$ for all $n \in \mathbb{Z}$ and all $t \geqslant 0$ because $\{\underline{u}_n\}$, $\underline{u}_n \equiv 0$ is a lower solution and $\{\bar{u}_n\}$, $\bar{u}_n \equiv 1$ is an upper solution of $\{u_n\}$. Therefore $\{u_n\}$, $\{v_n\}$, where $v_n \equiv u_n$, is the unique solution of

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \\ u_n &= P(v_n), \qquad v_n(0) = y_n, \qquad n \in \mathbb{Z}. \end{aligned} \tag{24}$$

Because of the continuous dependence of the solution of (24) on its initial condition y and on the function f, we conclude that $u_n(t)$ is nondecreasing and $u_n(\tau) = u_{n+1}(0)$. It remains to be shown that $\dot{u}_n(t) > 0$ for all $n \in \mathbb{Z}$ and all $t \in \mathbb{R}$. We already know that

$$\dot{u}_j(s) \geqslant 0 \qquad \text{for all} \quad j \in \mathbb{Z} \text{ and all } s \in \mathbb{R}. \tag{25}$$

Suppose that $\dot{u}_n(t) = 0$ for some $n \in \mathbb{Z}$ and some $t \in \mathbb{R}$. Then $\ddot{u}_n(t)$ exists and

$$\ddot{u}_n(t) = d(\dot{u}_{n-1}(t) + \dot{u}_{n+1}(t)). \tag{26}$$

Using (25) we conclude $\ddot{u}_n(t) \geqslant 0$. Since $\ddot{u}_n(t) > 0$ would lead to $\dot{u}_n(t - \varepsilon) < 0$ for sufficiently small $\varepsilon > 0$ therefore contradicting (25), we infer $\ddot{u}_n(t) = 0$. It follows then from (25) and (26) that $\dot{u}_{n-1}(t) = 0$ and $\dot{u}_{n+1}(t) = 0$. Hence $\dot{u}_n(t) = 0$ for all $n \in \mathbb{Z}$ by induction, which implies that the wave $\{u_n\}$ has zero speed in contradiction to $c = 1/\tau > 0$. This completes the proof of Theorem 1.

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